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The Nash Bargaining Problem Without Convexity†

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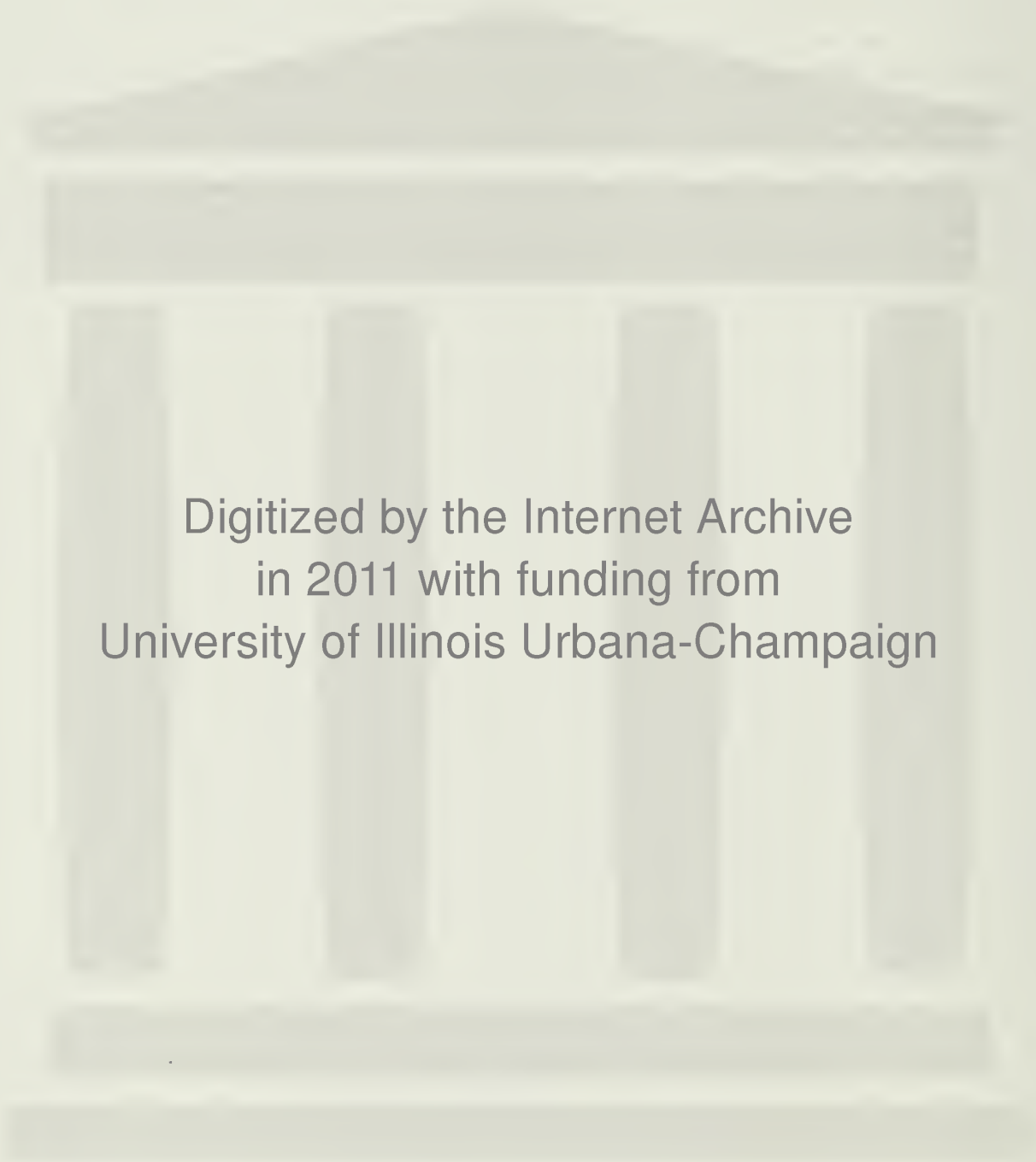
Abstract

An *n-person Bargaining Problem* consists of a pair (S, d) where $S \subset \Re^n$ is a set of feasible utility vectors which the players may obtain through cooperation, and the point $d \in S$, called the *disagreement point*, is interpreted as the utility that players receive if they fail to reach an agreement. Given a class of bargaining problems, Σ^n, a *solution* is a map that associates with each problem (S, d) in Σ^n a unique point in S. In this paper, we investigate the Nash solution on a domain of comprehensive but not necessarily convex problems. We argue that even with von Neumann–Morgenstern utilities, restricting attention to convex problems limits the application of the theory. Without convexity the Nash solution is not well defined. We propose a new solution called the Nash Extension. This solution coincides with the Nash solution when S is convex and is characterized by weak Pareto optimality, symmetry, scale invariance, continuity, and a new axiom, contraction monotonicity. Finally, we present a non-cooperative game that implements the Nash extension solution.



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1. Introduction

An *n-person bargaining problem* consists of a pair (S, d) where S is a non-empty subset of $\Re^n$, and $d \in S$. The set S is interpreted as the set of utility allocations that are attainable through joint action on the part of all n agents. If the agents fail to reach an agreement, then the problem is settled at the point d, which is called the *disagreement point*. A *bargaining solution* F, defined on a class of problems Σ^n, is a map that associates with each problem $(S, d) \in \Sigma^n$ a unique point in S. In the axiomatic approach to bargaining we start by specifying a list of properties (Pareto optimality, for example) that we would like a solution to have. If it can be shown that there is a unique solution that satisfies a given list of axioms, then the solution is said to be *characterized* this list.

It is common to restrict the domain to problems with convex feasible sets. However, bargaining problems can arise from a variety of political, social and economic situations. The requirement that S be convex seems to remove many important cases from consideration. For example, the image in utility space of a finite set of resource allocations will be a finite set of points, not a convex set. Or consider the bargaining problem associated with an economy in which strong externalities are present. It is quite likely that the feasible set of such a problem will be non-convex.

This restriction of domain is often justified by assuming that agents' preferences can be represented by von Neumann–Morgenstern utility functions. Feasible sets may then be convexified by running lotteries over the original elements. However, the proposition that agents satisfy the von Neumann–Morgenstern assumptions of preference relations are often rejected in the empirical literature (for example, see Machina(1984)), and serious questions of interpretation remain in any event. In particular, if a problem is settled at a lottery then the axioms that characterize the solution are only satisfied in expectation, and not, in general, by the actual utility allocations that agents walk away with after the lottery is held. Thus, even if we accept the von Neumann–Morgenstern assumptions on preferences, we must still decide whether the that axioms should be satisfied by the ex-anti or ex-post allocations.

This question is important if bargaining theory is viewed as a method of prescribing settlements to social distribution problems that are "fair" or "ethical". Under this interpretation, the axioms that characterize the solution employed are equivalent to a description of the ethical values of the society. For example, consider a society with two agents. Suppose that slavery is permitted in this society, and that if one agent is a slave and the other the master, the utility allocations are 0 and 10 respectively. If both agents are freemen, then each gets a utility allocation of 2. Also suppose that we are egalitarian enough to be believe in symmetry and Pareto optimality. Then if we allow lotteries, the problem is settled by flipping a fair coin to determine who is to be slave and who is to be master. However, would be difficult to argue that a society in which half the agents are slaves is egalitarian or even symmetric. In this case, the ex-post notion of fairness is the appropriate one. This can also be viewed as a choice between fairness of opportunity and fairness of result.[1] Allowing problems to be settled at lotteries is equivalent to deciding that outcomes med not satisfy out ethical notions. In many situations, this may be inappropriate and restrictive.

Following Zeuthan and Harsanyi, bargaining theory may also be used to help understand certain non-cooperative situations. Consider, for example, the case of mediator attempting to settle a labor dispute. Mediation is predicated on the belief that if agents are persuaded that a particular division is "fair", then they will voluntarily agree to coordinate their actions and accept the allocation as a settlement to their problem. Thus, the axioms summarize what the mediator believes the agents are likely to accept as fair. Here the choice of whether or not to allow settlements that are only ex-anti fair depends upon whether or not the agents can sign binding contracts to abide by the outcomes of lotteries. If they cannot, then any settlement that is not ex-post fair will break down after the lottery is held. Thus, lotteries must be rejected as possible settlements in such problems even if agents have von Neumann–Morgenstern preferences.

In this paper we concentrate on extending the Nash solution to the domain of com-

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[1] We thank Charles Kahn for this suggestion.

prehensive but not necessarily convex bargaining problems.[2] In section two we define the axioms and mathematical objects used in subsequent sections. In section three we propose a new solution called the *Nash extension* and give an axiomatic characterization. We discuss the reasons why the ordinary Nash solution is not well defined on this domain, and argue that our extension has several important properties that alternative generalizations in the literature fail to satisfy. In section four we describe a non-cooperative game that implements the new solution. This is a particularly important aspect of this paper given the Gibbard–Satherwaite impossibility theorem which tells us agents would not truthfully respond if they were asked to report their utility functions directly. The game requires agents to bid for first mover advantage in the manner of Moulin's(1984) game (which implements the Kalai–Smorodinsky solution). Subsequently, players can make offers in the manner of the Rubinstein–Binmore (1982) game (which implements the Nash solution). Our result is akin to that of Rubinstein and Binmore. We show that in the limit, the unique subgame perfect equilibrium payoff is exactly the utility allocation prescribed by our proposed solution to the cooperative game.

[2] In a companion paper, Conley and Wilkie(1989), we also study the Kalai–Smorodinsky and the egalitarian solutions on this domain.

2. Definitions and Axioms

We start with some definitions and formal statements of the axioms used in the - characterizations. Given a point $d \in \Re^n$, and a set $S \subset \Re^n$, we say S is *d-comprehensive* if $d \leq x \leq y$ and $y \in S$ implies $x \in S$.[3] The *comprehensive hull of a set* $S \subset \Re^n$, *with respect to a point* $d \in \Re^n$ is the smallest d-comprehensive set containing S:

$$comp(S;d) \equiv \{x \in \Re^n \mid x \in S \text{ or } \exists\, y \in S \text{ such that } d \leq x \leq y\}. \tag{1}$$

The *convex hull of a set* $S \subset \Re^n$ is the smallest convex set containing the set S:

$$con(s) \equiv \left\{ x \in \Re^n \mid x = \sum_{i=1}^{n+1} \lambda_i y_i \text{ where } \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \geq 0 \; \forall\, i, \text{ and } y_i \in S \; \forall\, i \right\}. \tag{2}$$

Let $\mathcal{C}$ denote the space of compact subsets of $\Re^n$. The Hausdorff distance $\rho : \mathcal{C} \times \mathcal{C} \to \Re$ is defined by,

$$\rho(S, S') \equiv \left\{ \max \left[\max_{x \in S'} \min_{y \in S} \parallel x - y \parallel ; \max_{x \in S} \min_{y \in S'} \parallel x - y \parallel \right] \right\} \tag{3}$$

where $\parallel \bullet \parallel$ is the Euclidean norm. A *closed* ϵ*-ball around* x is defined as:

$$B_\epsilon(x) \equiv \{z \in \Re^n \mid \parallel x - z \parallel \leq \epsilon\}. \tag{4}$$

Let $int(S)$ denote the *interior* of S, and $\partial(S)$ the boundary of S.
Define the *weak Pareto frontier* of S as:

$$WP(S) \equiv \{x \in S \mid y \gg x \text{ implies } y \notin S\}. \tag{5}$$

[3] The vector inequalities are represented by $\geq, >, and \gg$.

Define the *strong Pareto frontier* of S as:

$$P(S) \equiv \{x \in S \mid y \geq x \text{ implies } y \notin S\}. \tag{6}$$

A bargaining problem (S, d) is said to be *strictly comprehensive* if $S = comp(S, d)$ and $WP(S) \equiv P(S)$. Roughly, this means that S is d-comprehensive, and the Pareto frontier has no vertical or horizontal segments. The domain of bargaining problems considered in this paper is Σ_c^n. This is defined as the class of pairs (S, d) where $S \subset \Re^n$ and $d \in \Re^n$ such that:

A1) S is compact.

A2) S is d-comprehensive.

A3) There exists $x \in S$ and $x \gg d$.

This differs from the usual domain, which we denote Σ_{con}^n, in that we do not assume that the set of feasible utility allocations is convex. A *bargaining solution*, F, is a function from Σ_c^n to $\Re^n$ such that for each $(S, d) \in \Sigma_c^n$, $F(S, d) \in S$.

A list of axioms used in this paper follows. Readers familiar with axiomatic bargaining theory may wish to skip to the definition of Compromise Monotonicity. All bargaining problems mentioned below are assumed to be elements of Σ_c^n.

Pareto-Optimality (W.P.O.): $F(S, d) \in WP(S)$.

Contraction Independence (C.IND)[4]: If $S' \subset S$, $d' = d$, and $F(S, d) \in S'$, then $F(S', d') = F(S, d)$.

A *permutation operator*, π, is a bijection from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$. Π^n is the class of all such operators. Let $\pi(x) = (x^{\pi(1)}, x^{\pi(2)}, \ldots, x^{\pi(n)})$.[5] and $\pi(S) = \{y \in \Re^n \mid y = \pi(x) \text{ for some } x \in S\}$.

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[4] C.IND was introduced by Nash(1950) with the label "Independence of Irrelevant Alternatives".

[5] Superscripts stand for the components of a vector

Symmetry (SYM): If for all permutation operators $\pi \in \Pi^n$, $\pi(S) = S$ and $\pi(d) = d$, then $F^i(S,d) = F^j(S,d)\ \forall\ i,j$.

An *affine transformation on* $\Re^n$ is a map, $\lambda : \Re^n \to \Re^n$, where $\lambda(x) = a + bx$ for some $a \in \Re^n, b \in \Re^n_{++}$. Λ^n is the class of all such transformations. Let $\lambda(S) = \{y \in \Re^n \mid y = \lambda(x), x \in S\}$.

Scale Invariance (S.INV): $\forall\ \lambda \in \Lambda^n$, $F(\lambda(S), \lambda(d)) = \lambda(F(S,d))$.

Continuity (CONT): For all sequences $\{(S^\nu, d)\}_{\nu=1}^{\infty}$, if $\rho(S, S^\nu) \to 0$, then $F(S^\nu, d) \to F(S,d)$.

The *Ideal Point* of a problem (S,d) is defined as:

$$a(S,d) \equiv (\max_{\substack{x \in S \\ x \geq d}} x^1, \max_{\substack{x \in S \\ x \geq d}} x^2, \ldots, \max_{\substack{x \in S \\ x \geq d}} x^n). \tag{7}$$

Restricted Monotonicity (R.MON): If $S \subset S'$, $d = d'$, and $a(S,d) = a(S',d')$, then $F(S',d') \geq F(S,d)$.

The *Compromise Point with respect to a solution F of a problem* (S,d), is defined as:

$$c^F(S,d) \equiv F(con(S),d). \tag{8}$$

Now *Compromise Monotonicity* is formally defined:

Compromise Monotonicity (C.MON): If $S' \subset S$, $d' = d$, and $c^F(S,d) \in con(S')$, then $F(S,d) \geq F(S',d')$.

Recall the motivation behind R.MON. We imagine each agent submitting a claim to level of utility. If it is possible to satisfy all the claims then we do so. If not then we take the vector of claims as an "ideal point", a point we would like to achieve but cannot reach. The actual solution must be feasible but should somehow take this ideal point into account. In particular, suppose we have a bargaining problem that undergoes an expansion of the feasible set (perhaps because of economic growth), in a way that leaves the ideal point and the disagreement point unchanged. Then the solution to the larger problem should leave all the agents at least as well off as they were at the solution to the original problem.

In short, R.MON is an axiom which gives a minimal property that the compromise away from the ideal point should satisfy.

The motivation for C.MON is similar. Instead of having agents submit claims, we imagine each agent proposing a feasible allocation as a settlement. If all the proposals agree, then the problem is settled. If not, then we imagine agents agreeing to "meet each other halfway".[6] If the feasible set is convex, then this compromise is feasible. If not, then we can think of the solution to the convex hull of the feasible set as being an "ideal compromise" which is not attainable. The actual solution must be feasible but should somehow take this compromise point into account. In particular, suppose we have a bargaining problem that undergoes an expansion of the feasible set in a way that leaves the compromise point and the disagreement point unchanged. Then the solution to the larger problem should leave all the agents at least as well off as they were at the solution to the original problem. And so, C.MON is an axiom that gives a minimal property that the compromise away from the ideal compromise should have.

[6] Or to make some more general compromise, meeting each other one third of the way along the line between the first and the second agents proposals for example. Different vectors of compromise weights might reflect different levels of bargaining power or toughness.

3. The Solutions

In his 1950 paper, Nash considered the domain Σ^n_{con} of convex problems. He proposed the following solution:

$$N(S,d) \equiv \left\{ \underset{\substack{x \in S \\ x \geq d}}{\operatorname{argmax}} \prod_{i=1}^{n} (x_i - d_i) \right\}, \tag{9}$$

and demonstrated that it is the uniquely characterized by W.P.O, SYM, S.INV, and C.IND. Kaneko(1980) offers a characterization of the direct generalization of Nash solution on the domain of "regular" bargaining problems. This domain is a superset of Σ^n_c. However, Kaneko's solution is not single-valued and is only upper-hemicontinuous. Thus his solution does not give an unambiguous recommendation of how to solve bargaining problems. In addition, small changes in the problem can lead to large changes in the outcome. This is the justification for Nash's inclusion of single-valuedness and continuity in his list of minimal requirements that bargaining solutions ought to satisfy. Herrero's(1989) generalization suffers from similar problems. She defines a generalization on the strictly comprehensive two person domain which gives outcomes that are a superset of the outcomes recommended by Kaneko's solution. Her solution is also multi-valued and satisfies a form of lower-hemicontinuity. Foster and Vohra(1988) give a characterization of the Nash solution on a domain in which the feasible set is not necessarily convex, but in which the maximizer of the Nash product is assumed to be unique. It is not clear what class of economic or political problems would give rise to such a domain. A natural suggestion would be to define a new solution by taking a selection from the set of maximizers of the Nash product. Unfortunately, it is impossible to do this in a way that satisfies the properties enjoyed by N on the convex domain. Obviously, no such selection satisfies SYM. In addition, it can be shown that *any* such selection must also fail to satisfy the axioms CONT and C.IND.

Thus, we need a solution that is continuous and single valued, coincides with the Nash solution if the problem is convex, and approximates the Nash solution otherwise. The Nash extension solution meets all of these requirements. We construct the Nash extension as

follows: First define the mapping $L : \Sigma_c^n \to \Re^n$ as:

$$L(S,d) \equiv con\Big(N\big(con(S),d\big),d\Big). \tag{10}$$

$L(S,d)$ is the line segment connecting the disagreement point d, to the Nash solution of the problem composed of the convex hull of S, and d. Now we define the solution NE:

$$NE(S,d) \equiv \{\max x \mid x \in L(S,d) \cap S\}, \tag{11}$$

Figure 1: The Nash extension solution

where max indicates the maximal element with respect to the partial order on $\Re^n$. The construction of NE is illustrated in figure 1. The point $NE(S,d)$ is the intersection of the weak Pareto frontier of S and the line segment connecting the disagreement point and compromise point under the Nash solution the problem (S,d). Obviously, NE coincides with N on the domain of convex problems. To see that the NE solution is single valued,

notice that L is a non-empty, compact-valued correspondence. Then since L is also a line segment, its maximal element exists and is unique. Thus, NE is non-empty and single-valued on Σ_c^n. Lemma 1 shows that the NE solution is continuous.

Lemma 1. *NE is continuous on Σ_c^n.*

Proof/

Let $S^\nu \to S$. We begin by showing that *con* is a ρ-continuous correspondence. To see this, suppose that for any given $\epsilon > 0$, $\rho(S, S^\nu) \leq \epsilon$. Then for any $y = \sum_{i=1}^{n+1} \alpha_i x_i \in con(S)$ there is a $y^\nu = \sum_{i=1}^{n+1} \alpha_i x_i^\nu \in con(S^\nu)$ such that $y \in B_\epsilon(y^\nu)$. By reversing the argument, we also find that for any $y^\nu \in con(S^\nu)$ there is a $y \in con(S)$ such that $y^\nu \in B_\epsilon(y)$. Thus if $\rho(S, S^\nu) \leq \epsilon$ then $\rho(con(S), con(S^\nu)) \leq \epsilon$, and so *con* is ρ-continuous.

Therefore, since *con* is continuous, and N is continuous on $\sum_{con}^n$, the composition map c^N, where $c^N(S, d) \equiv N(con(S), d)$, is continuous by Hildenbrand(1974) proposition B.7. We conclude that if $S^\nu \to S$, then $c^N(S^\nu, d) \to c^N(S, d)$.

By Definition, $NE(S, d) \in L(S, d)$. So $NE(S, d) = (1 - \lambda^*)d + \lambda^* c^N(S, d)$ for some $\lambda^* \in [0, 1]$. Also, for each S^ν, $NE(S^\nu, d) = (1 - \lambda^\nu)d + \lambda^\nu c^N(S^\nu, d)$ for some $\lambda^\nu \in [0, 1]$. Notice that the sequence $\{\lambda^\nu\}$ is drawn from the compact set $[0, 1]$. Thus, given any sequence of sets $\{S^\nu\}$ converging to S, if it can be shown for every convergent subsequence $\{\lambda^{\nu_k}\}$ of $\{\lambda^\nu\}$ that $\lambda^{\nu_k} \to \lambda^*$ then the lemma is proven. Suppose not. Then there are two cases:

1. Suppose first that for some subsequence $\{S^{\nu_k}\}$, $\lambda^{\nu_k} \to \hat{\lambda}$ and $\hat{\lambda} > \lambda^*$. Then the definition of NE implies $(1 - \hat{\lambda})d + \hat{\lambda} c^N(S, d) \equiv \hat{x} \notin S$. Thus the sequence $\{NE(S^{\nu_k})\}$ converges to a point not in S, contradicting the hypothesis $S^{\nu_k} \to S$.
2. Now suppose that for some subsequence $\{S^{\nu_k}\}$ that $\lambda^{\nu_k} \to \hat{\lambda}$ and $\hat{\lambda} < \lambda^*$. Then $(1 - \hat{\lambda})d + \hat{\lambda} c^N(S, d) \equiv \hat{x} \ll NE(S, d)$. Additionally, the existence of a point S that strictly dominates d implies that $d \ll \hat{x}$. Hence by the d-comprehensiveness of S, $\hat{x} \in int(S)$. Thus there exists $\epsilon > 0$ and ν_1 such that for $\nu > \nu_1$, $B_\epsilon(\hat{x}) \subset S^{\nu_k}$ and $NE(S, d) \notin B_\epsilon(\hat{x})$. Since $c^N(S^{\nu_k}, d) \to c^N(S, d)$, there exists ν_2 such that $\nu > \nu_2$ implies $L(S^{\nu_k}, d) \cap B_\epsilon(\hat{x}) \neq \emptyset$. Now, for each ν_k let $y^{\nu_k} = \max\{L(S^{\nu_k}, d) \cap B_\epsilon(\hat{x})\}$. Let $\nu' \equiv \max\{\nu^1, \nu^2\}$. Clearly for $\nu_k > \nu'$, y^{ν_k} exists and $y^{\nu_k} \in \partial B_\epsilon(\hat{x}) \cap S^{\nu_k}$. However, by

hypothesis $NE(S^{\nu_k}, d) \to \hat{x}$, so there exists ν'' such that $\nu_k > \nu''$ implies $NE(S^{\nu_k}, d) \in int(B_\epsilon(\hat{x}))$. Then for $\nu_k > \max\{\nu', \nu''\}$, we have $y^{\nu_k} \in S^{\nu_k} \cap L(S^{\nu_k}, d)$ and $y^{\nu_k} \gg NE(S^{\nu_k}, d)$, contradicting the definition of NE. Hence for every subsequence of S^ν, we have that $\hat{\lambda} = \lambda^*$. Therefore, $\hat{x} = NE(S, d)$.

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Our main result is a characterization of the new solution NE.

Theorem 1. *A solution on Σ_c^n satisfies W.P.O, S.V, SYM, C.MON, and CONT if and only if it is the Nash extension.*

Proof/

(a) First it is shown that the NE solution satisfies the axioms.

W.P.O: Let $x = NE(S, d)$. Assume there exists $y \in S$ such that $y \gg x$. Then since S is d-comprehensive there exists $z \in L(S, d) \cap S$ such that $z \gg x$. However, this contradicts the hypothesis $x = NE(S, d)$.

S.INV: Let $(S, d) \in \Sigma_c^n$ and $\lambda \in \Lambda^n$ be any affine transformation. Since $con(\lambda(S)) = \lambda(con(S))$, and N satisfies S.INV on Σ_{con}^n, we conclude that $N(\lambda(con(S)), \lambda(d)) = \lambda(N(con(S), d))$. Thus $L(\lambda(S), \lambda(d)) = \lambda(L(S, d))$. Therefore, $\max\{L(\lambda(S), \lambda(d)) \bigcap \lambda(S)\} = \max\{\lambda(L(S, d) \cap S)\} = \lambda(NE(S, d))$, as required.

SYM: Let (S, d) be a symmetric problem. Then $(con(S), d)$ is also a symmetric problem. Since N satisfies SYM on Σ_{con}^n, $N(con(S), d)$ is a point of equal coordinates. But so is d, and so all elements $L(S, d)$ are points of equal coordinates. Consequently, $NE(S, d) \in L(S, d)$ is symmetric.

C.MON: Let $(S, d), (S', d')$ be such that; $S \subset S'$, $d = d'$ and $c^{NE}(S', d') \in con(S)$. Then $N(con(S'), d') = NE(con(S'), d') = c^{NE}(S', d')$, and therefore $N(con(S'), d') \in con(S)$. Since $con(S) \subset con(S')$, and N satisfies C.IND on Σ_{con}^n, $N(con(S), d) = N(con(S'), d')$. Furthermore since $d = d'$ by hypothesis, $L(S, d) = L(S', d')$. Therefore $S \subset S'$ implies $NE(S, d) \leq NE(S', d')$, as required.

CONT: See lemma 1.

(b) Conversely let F be a solution on Σ_c^n satisfying the five axioms, and consider any

problem (S,d). By S.INV, we can set $d = 0$ and $N(con(S),d) = (1,1,\ldots,1) \equiv c$. Then $NE(S,d) = (\alpha,\ldots,\alpha) \equiv x$ for some $\alpha > 0$. We distinguish two cases:

Case 1) $S \subset \Re^n_+$.

Let the sets T and V be defined as follows,

$$T \equiv concomp((n,0,\ldots,0),(0,n,\ldots,0),(0,\ldots,n);d) \tag{12}$$

$$V \equiv T \setminus \{x + \Re^n_{++}\}. \tag{13}$$

Since $c = N(con(S),d)$, the hyperplane defined by $\sum_{i=1}^n x_i = n$ supports $con(S)$ at c. Hence $S \subset T$. Also, since F satisfies W.P.O, and S is comprehensive, $z \in \{x + \Re^n_{++}\}$ implies that $z \notin S$. Thus $S \subseteq V$.

Now, since $(V,0)$ is a symmetric problem, and x is the only symmetric point in $WP(V)$, by W.P.O. and SYM, $F(V,d) = x$. Also, since c is the only symmetric point in $WP(T)$, by W.P.O. and SYM, $F(T,d) = e$. But $con(V) = T$, and so $c^F(V,d) = c$. Therefore, since $S \subseteq V$ and $c^F(V,d) = c \in con(S)$, by C.MON, $F(S,d) \leq F(V,d) = x$. There are two possibilities.

i) $x \in P(S)$. Then by W.P.O, $F(S,d) = x = NE(S,d)$ and the proof is complete.

ii) $x \notin P(S)$. Then consider the sequence of problems $\{(V^\nu;0)\}$ and $\{(S^\nu;0)\}$ defined by:

$$V^\nu \equiv \left\{ V \bigcup comp\left[\frac{1}{\nu}c + (1-\frac{1}{\nu})x;0\right]\right\} \tag{14}$$

$$S^\nu \equiv \left\{ S \bigcup comp\left[\frac{1}{\nu}c + (1-\frac{1}{\nu})x;0\right]\right\}. \tag{15}$$

Since V^ν is symmetric and $d = 0$, by W.P.O. and SYM, $F(V^\nu,0) = (\alpha+\frac{1}{\nu},\ldots,\alpha+\frac{1}{\nu}) \equiv x^\nu$. Since $S^\nu \subseteq V^\nu$, $c^F(V^\nu,0) = c$ and $c \in con(S^\nu)$, by C.MON and W.P.O. we have

$F(S^\nu, d) = F(V^\nu, d) = x^\nu$. But since $S^\nu \to S$, by CONT $F(S^\nu, d) \to F(S, d)$. Thus since $x^\nu \to x$, we conclude that $F(S, d,) = x = NE(S, d)$.

Case 2) $S \not\subset \Re^n_+$.

Let $V' \equiv \{V \bigcup_{\pi \in \Pi^n} \pi(S)\}$. Since V' is symmetric, and $x \in WP(V')$, we can replace $(V, 0)$ above with $(V', 0)$ and replicate argument given for Case 1.

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The Nash extension solution has two independently interesting properties. As we noted earlier, on the domain Σ^n_{con} it coincides with the Nash solution. In fact, on the convex domain, our new axiom, C.MON, along with W.P.O. imply C.IND.

Now consider a domain of "anti-convex problems", Σ^n_{a-con}. This is the class of probems that satisfy $A1 - A3$ and the following two additional requirements.

A4) For all $x \in S, x \geq d$.

A5) $\{x \in \Re : x \geq y \; \forall \; y \in S\}$ is convex.

Anti-convex problems are "concave upward" instead of "concave down" and so might represent economies with increasing marginal returns instead of diminishing marginal returns. In this seanse, convex and anti-convex economies are polar opposits. It is not hard to see that on this domain, W.P.O, S.INV, SYM, and and C.MON imply R.MON. Recall that the definition of the Kalai–Smorodinsky solution $K : \Sigma^n_{a-con} \subset \Sigma^n_{con} \to \Re$:

$$K(S, d) \equiv \max[x \in S \mid x \in con(a(S, d), d).$$

We show in Conley and Wilkie(1989) that the Kalai-Smorodinsky solution is characterized by W.P.O, S.INV, SYM, and R.MON, on Σ^n_c. thus, on Σ^n_{a-con} the Nash extension coincides with the Kalai-Smorodinsky solution. We show this formally in the lemma 2.

Lemma 2. $\forall \, (S, d) \in \Sigma^n_{a-con}, NE(S, d) = K(S, d)$.

Proof/

Let $(S, d) \in \Sigma^n_{a-con}$, Apply an affine transformation $\lambda \in \Lambda$ to (S, d) such that $\lambda(d) = 0$, and $a(\lambda(S), 0) = (1, 1, \ldots, 1)$. Then $K(\lambda(S), 0)$ is the largest element of $\lambda(S)$ that is on the 45° line between $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$.

Clearly, by A4 and A5, $con(\lambda(S))$ just the unit simplex. Thus, $c^N(\lambda(S),0) = N(con(\lambda(S),0 =$ $(1/n, 1/n, \ldots, 1/n)$ And so $NE(\lambda(S),0)$ is the largest element on $\lambda(S)$ that is on the 45° line between $(0,0,\ldots,0)$ and $(1/n,1/n,\ldots,1/n)$. Therefore, by S.INV, $NE(S,d) = K(S,d)$.

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Thus,the Nash extension solution can be seen as hybrid of the Nash and the Kalai-Smorodinsky solutions that coincides with Nash when, in our language, all compromises are possible, and the Kalai-Smorodinsky solution when no efficient and individually rational compromises are possible. In a similar manner the axiom C.MON, can be seen as a hybrid of C.IND and R.MON.

4. Implementing the Nash Extension Solution

In this section we present a non-cooperative game that implements the Nash extension solution. The problem we have in mind is exactly that studied by Moulin(1984). We wish to construct a non-cooperative game in which agents' utility functions are private information and which gives utility allocations recommended by the Nash extension solution to the bargaining problem as the unique equilibrium. This is strictly non-cooperative problem. The agents act purely out of self interest, and can communicate only through the mediator and never directly. Formally, the result we obtain is similar to those of Binmore *at al*(1986) and Rubinstein(1982), and Herrero(1989). That is,"the limit of the (unique) subgame perfect equilibrium payoffs, as the discount parameter goes to one, is the utility allocation prescribed by the Nash extension solution."

We consider the case of two agents, 1 and 2, who have available a set of real economic alternatives, A, with a distinguished element, $a^d \in A$, as the disagreement alternative. The agents have utility functions u_1 and u_2 defined over A which yields the bargaining problem (S, d), where $S \subset \Re^2_+$ is the image of A under the map (u_1, u_2) and d is the image of a^d. From now on we normalize utilities so that $d = (0, 0)$. We will assume that S is strictly comprehensive and so there exists a continuous, strictly decreasing function $f : \Re_+ \to \Re_+$, where $f(x) = max\{y \in \Re_+ | (x, y) \in S\}$. We will suppose that the preferences satisfy the von-Neumann Morgenstern axioms and so we may extend u_i to $\mathcal{M}(\mathcal{A})$, the space of lotteries defined over A. We denote by $M \subset \Re^2_+$ image of $\mathcal{M}(\mathcal{A})$ under the maps u_1, u_2. The pair (M, d) is also a bargaining problem, but with a convex feasible set.

Recall the definition of the Rubinstein alternating offer game. Time is discrete. In each time period, t, one player gets to propose an allocation in S to the other who can either accept or refuse. If the second player refuses, the feasible set is discounted at rate δ, and in the next period she makes an offer. If this offer is refused, the feasible set is again discounted, and player one makes an offer. The game alternates until an offer is accepted. If no offer is ever accepted, the players obtain the disagreement point.

Let $\delta \in (0, 1)$ be given, and define the game $\Gamma(\delta)$ as follows. The game has three

stages. Stage 1:

Each player chooses and number from the set $P_i = [0, 1]$. If $p_1 > p_2$ then 1 proposes an allocation $a_1 \in \mathcal{A}$. Player 2 announces *"Yes"* or *"No"*.

If player 2 announces *"Yes"* the game ends with payoffs $u_1 = u_1(a_1), u_2 = u_2(a_1)$. If player 2 announces *"No"*, then the game proceeds to Stage 2.

If $p_1 < p_2$ then the roles are reversed, player 2 gets to propose an allocation.

If $p_1 = p_2$ then a fair coined is tossed to decide who proposes an allocation.

Stage 2:

If $p_1 > p_2$ then with probability $1 - p_2$ the bargaining process ends and the outcome is $u_1 = u_2 = 0$. With probability p_2, the game proceeds to Stage 3.

If $p_1 \leq p_2$ then with probability $1 - p_1$ the bargaining process ends and the outcome is $u_1 = u_2 = 0$. With probability p_1, the game proceeds to Stage 3.

Stage 3:

If $p_1 > p_2$ then player 2 is chosen as the leader in Rubinstein's alternating offer game with discount factor δ applied between offers. Player 2 proposes an element in $\mathcal{M}$. Player 1 can accept or reject the offer. If he accepts, then the offer is implemented with the mediator enforcing the outcome, including any required randomization. If player 1 rejects 2's offer, then from the next time period, play proceeds with 1 making a proposal from $\mathcal{M}$.

If $p_1 < p_2$ then the player's roles above are reversed.

If $p_1 = p_2$, then a fair coin is tossed to decide who makes the first offer.

We begin with some well known facts.

Fact 1. In the Rubinstein sub-game, for each value of $\delta < 1$, there exists a unique equilibrium payoff pair, $(x_1^o(\delta), x_2^r(\delta))$ when 1 has the first offer and 2 replies. $(x_1^o(\delta), x_2^r(\delta))$ when 2 has the first offer and 1 replies.

Fact 2. $Lim_{\delta \to 1}(x_1^o(\delta), x_2^r(\delta)) = Lim_{\delta \to 1}(x_1^o(\delta), x_2^r(\delta)) = N(M, d)$.

We now introduce some notation.

Let $(x_1(\delta), x_2(\delta)) = (x_1^o(\delta), x_2^o(\delta))$. Let $\hat{p}(\delta)$ be defined as the maximal value of p such that $(p \cdot \delta x_1(\delta), p \cdot \delta x_2(\delta)) \in S$.

If $\hat{p}(\delta) \leq 1$, let $(e_1(\delta), e_2(\delta)) = (\hat{p} \cdot x_1(\delta), \hat{p} \cdot x_2(\delta))$. If $\hat{p}(\delta) > 1$ let $(e_1(\delta), e_2(\delta)) = (\frac{1}{2}(f^{-1}(\delta x_2^o) + \delta x_1^o), \frac{1}{2}(f(\delta x_1^o) + \delta x_2^o))$.

Lemma 1. *For all $\delta < 1$ the game $\Gamma(\delta)$ has a unique subgame perfect equilibrium payoff, $(u_1^*, u_2^*) = (e_1(\delta), e_2(\delta))$.*

Proof/

As δ is fixed in the proof we drop δ when it appears as an argument. We distinguish two cases.

Case 1, $\hat{p}(\delta) > 1$.

If $p_1 > p_2$ and the game reaches Stage 3, then from Fact 2 the unique subgame perfect equilibrium continuation payoffs are $(\delta x_1^r, \delta x_2^o)$. Now suppose $p_1 < 1$. Then if $p_2 < p_1$, in stage one, 1 makes an offer to 2. By subgame perfection, 2 will accept an offer, a', if and only if $u_2(a') \geq p_2 \delta x_2^o$. By hypothesis, as $\hat{p}(\delta) > 1$, then $x \in int(S)$, thus there exists an allocation a such that $u_1(a) = f^{-1}(p_2 \delta x_2^o) > p_2 \delta x_1^r$, and $u_2(a) = p_2 \delta x_2^o$. Thus player 1 will not propose an outcome a', that yields $u_2(a') > p_2 \delta x_2^o$. If $p_2 = p_1$, then the previous argument holds if 1 is chosen to offer, and the symmetric argument if 2 is chosen. Thus the utility 2 obtains is $\frac{1}{2}(f(p_1 \delta x_1^o) + p_1 \delta x_2^o))$. If $p_2 > p_1$ then 2 proposes to player 1. By subgame perfection, 1 will accept an offer, a', if $u_1(a') \geq p_1 \delta x_1^o$. As $p_1 < \hat{p}(\delta)$ there exists an outcome a such that $u_2(a) = f(p_1 \delta x_1^o) > f(\delta x_1^o) = x_2^o > p_2 \delta x_2^o$ and $u_2(a') > p_2 \delta x_2^o$. Thus it is best response for 2 to set $p_2 > p_1$. The symmetric argument holds for player 1 if $p_2 < 1$. Thus in any equilibrium it must be that $p_1 = p_2 = 1$. However then the unique subgame perfect equilibrium payoffs are, $(\frac{1}{2}(f^{-1}(\delta x_2^o) + \delta x_1^o), \frac{1}{2}(f(\delta x_1^o) + \delta x_2^o)) \equiv (e_1(\delta), e_2(\delta))$.

Case 2, $\hat{p}(\delta) \leq 1$.

Suppose $p_1 < \hat{p}(\delta)$. If $p_2 < p_1$, in stage one, 1 makes an offer to 2. By hypothesis, as $p_2 < \hat{p}(\delta)1$, then $p_2(x_1^r, x_2^o) \in int(S)$. Thus there exists an allocation a such that $u_1(a) = f^{-1}(p_2 \delta x_2^o) > p_2 \delta x_1^r$. A similar argument to Case 1 above, shows that the best response for 2 to set $p_2 > p_1$. The symmetric argument holds for player 1 if $p_2 < \hat{p}(\delta)$. Thus it must be that p_1 and $p_2 \geq \hat{p}(\delta)$.

If $p_1 > \hat{p}(\delta)$ and $p_2 > p_1$, then in stage one, 2 proposes an allocation to 1. By subgame perfection, 1 will reject any offer a with $u_1(a) < p_1 \delta x_1^o(\delta)$. By construction, as $p_1 > \hat{p}(\delta)$,

$f(p_1\delta x_1^o(\delta)) < p_1\delta x_2^o(\delta)$. Furthermore $x_2^r(\delta) < x_2^o(\delta)$. Thus the highest payoff that 2 can attain if $p_2 \geq p_1$ is the maximum of $f(p_1\delta x_1^o(\delta))$ and $p_1\delta x_2^r(\delta)]$. However, by hypothesis there exists a $p_2' < p_1$ that ensures 2 a payoff of a least $p_2'\delta x_2^o(\delta) > Max[f(p_1\delta x_1^o(\delta)), p_1\delta x_2^r(\delta)]$. Thus if $p_1 > \hat{p}(\delta)$, then player 2's best response must be $p_2 < p_1$. The symmetric argument holds for player 1 if $p_2 > \hat{p}(\delta)$. Thus in any equilibrium it must be that $p_1 = p_2 = \hat{p}(\delta)$. However then the unique subgame perfect equilibrium payoffs are, $(\hat{p} \cdot x_1(\delta), \hat{p} \cdot x_2(\delta)) \equiv (e_1(\delta), e_2(\delta))$.

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We now present the main result of this section.

Theorem 1. *The limit as $\delta \to 1$ of the subgame perfect equilibrium payoffs of $\Gamma(\delta)$ is exactly the Nash-extension solution to the bargaining problem (S, d).*

Proof/

From lemma 1 we have that $\forall\, \delta\ E(\Gamma(\delta)) = (e_1(\delta), e_2(\delta))$. ¿From Fact 2 we know that $Lim_{\delta \to 1}(x_1^o(\delta), x_2^r(\delta)) = Lim_{\delta \to 1}(x_1^o(\delta), x_2^r(\delta)) = N(M, d)$ Then we have that $x_1^o(\delta) \to x_1^r$ and $x_2^o(\delta) \to x_2^r$. Furthermore $\delta S \to S$, and therefore $\delta M \to M$. Thus by the definition of $NE(S, d)$, $\hat{p}(\delta) \to \lambda) \leq 1$. Therefore, by the continuity of multiplication on $\Re$. $E(\Gamma(\delta)) \to \lambda N(M, d)$. However by definition, $\lambda N(M, d) = NE(S, d)$.

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5. Conclusion

Since Nash's pioneering treatment of the bargaining problem, most authors have maintained the assumption of convexity of the feasible set. In this paper we have dispensed with convexity. We proposed a new solution, the Nash Extension, and demonstrated that it retains several of the desirable features of the Nash solution on the domain of comprehensive problems, while coinciding with the Nash solution when the problem is convex.

Our main result is a characterization of the new solution, employing a new axiom, Ethical Monotonicity. Additionally we demonstrate that axiomatic characterizations of several well known solutions can be extended to the domain of problems that are merely comprehensive. This does not seem to be a very strong restriction on the domain since it is implied by an assumption of freely disposable utility.

This work suggests that the assumption of a convex feasible set is not essential for any Monotone Path Solution. Since any Monotone Path Solution is well-defined on the domain of comprehensive problems any characterization found on the domain of convex problems should be easy to adapt. This class of solutions is discussed and axiomatized Thomson (1986), pp52-57. A second class of solutions that are well defined on the domain of convex problems is the class of strictly concave social welfare functions. The Nash solution is the most widely known of these. The class of solutions represented by an additively separable social welfare function has recently been axiomatized by Lensberg(1988). It would be of interest to see if our method of constructing the Nash Extension could be employed to define a new solution or solution class on the domain of comprehensive problems, which coincides with the selection of the given social welfare function or class of functions, when the problem is convex. A characterization could then be attempted using C.MON or some similar axiom.

We close by remarking that it may also be of interest to study more general domains. For example, suppose agents cannot necessarily dispose of utility freely, but they can "agree some of the time and disagree some of the time," then the we have a domain of problems which the feasible sets are star-shaped with respect to the disagreement point.

In this note we have provided a game that implements the Conley-Wilkie solution for non-convex bargaining problems, in the same manner that the Rubinstein-Binmore alternating offer game implements the Nash solution for convex problems. Recently a game which exactly implements the nash solution, rather than as a limiting case, has been developed by Howard [1988] It would be interesting to see if we can construct a game that exactly implements the Conley-Wilkie solution.

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